Loose simplicial objects

A useful shape for homotopy limits in (holomorphic) geometry

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Introduction

Motivation

Two stories with a common picture

1. **Theorem.** If $X^\star_\bullet$ is a Reedy fibrant cosimplicial simplicial set then the Bousfield–Kan map is a weak equivalence of simplicial sets: $\mathbf{holim}\, X^\star_\bullet \xrightarrow{\sim} \mathrm{Tot}\, X^\star_\bullet$

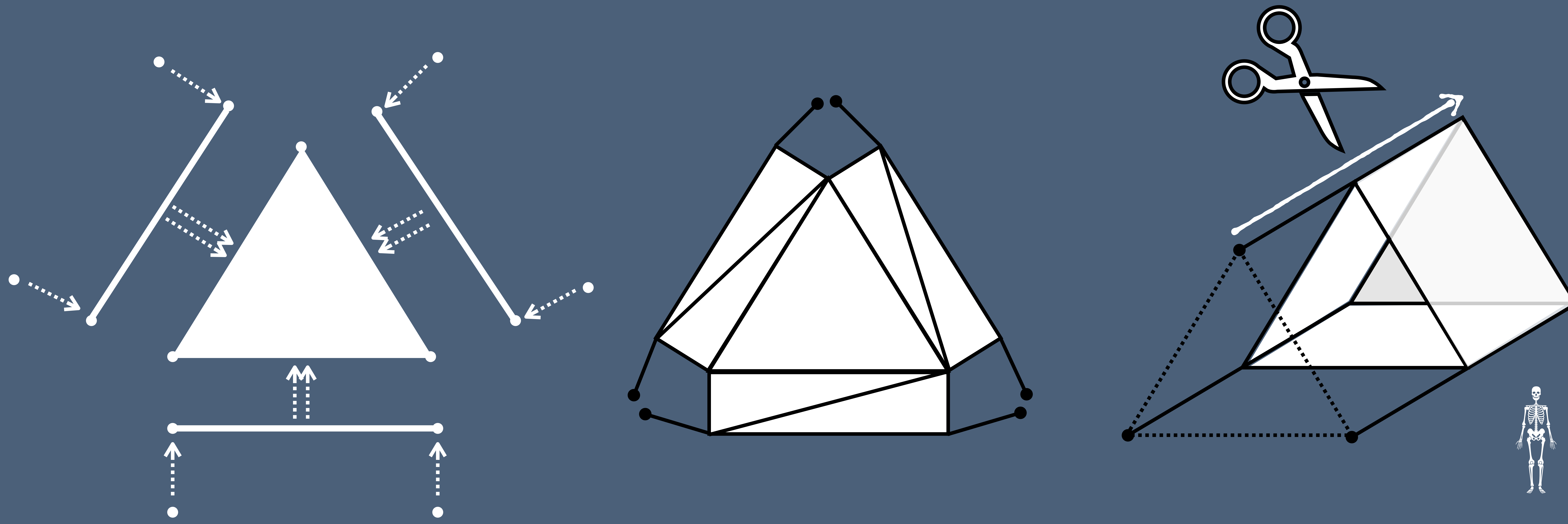
   $\longrightarrow$ **Proof. [Bousfield–Kan]** We can "semi-strictify" the fat simplex to the simplex

2. **Theorem.** If $\mathscr{F}$ is a complex of analytic sheaves with coherent cohomology, then there exists a (very nice) complex $\mathscr{R}^\star_\bullet$ of locally free sheaves on the Čech nerve such that $\mathscr{R}^\star_\bullet \xrightarrow{\sim} i^*\mathscr{F}^\star_\bullet$

   $\longrightarrow$ **Proof. [Green–H]** We can "semi-strictify" the barycentric subdivision to the pair subdivision

The loose/extruded simplices

Defined via hand-waving (for now)



Consequences

Work in progress (with Cheyne Glass) — conjectural

- We can define a spectrum of constructions that lie in between totalisation and the homotopy limit: $\mathrm{Tot} = \mathrm{Tot}_0 \hookrightarrow \mathrm{Tot}_1 \hookrightarrow \mathrm{Tot}_2 \hookrightarrow \ldots \hookrightarrow \mathrm{Tot}_\infty = \mathbf{holim}$
- In "nice" cases (e.g. Reedy fibrant), the above spectrum fully collapses; we might be able to define a spectrum of "niceness" for which it partially collapses
- Just as *twisting cochains* are the $\mathrm{Tot} = \mathrm{Tot}_0$ of a simplicial presheaf, we can realise *simplicial twisting cochains* as exactly the Tot_1 of the same simplicial presheaf. (**Corollary.** They are actually equivalent)
- ... a better understanding of $\mathrm{D}^{\mathrm{b}}(\mathsf{Coh}(X)) \hookrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathsf{Sh}(X))$?

Coherent sheaves

Generality

What we won't talk about

- Many things actually hold for arbitrary (locally) ringed spaces, see

  H and Zeinalian, "Simplicial presheaves of Green complexes and twisting cochains", (2023) arXiv:2308.09627

- All of the work is inspired by the complex-analytic (i.e. holomorphic) case:
  - X a (paracompact) holomorphic manifold
  - $\mathscr{O}_X$ the sheaf of holomorphic functions
- In particular, no assumptions about algebraicity (e.g. Kähler, projective)

Locally free complex-analytic sheaves

A sheaf-theoretic version of holomorphic vector bundles

- Complex-analytic manifold $(X, \mathcal{O}_X)$ with (trivialising) (Stein) cover $\mathcal{U} = \{U_\alpha\}$
- Locally free sheaf $E \to X$ of rank r with trivialisations $\varphi_\alpha\colon E|U_\alpha \xrightarrow{\sim} (\mathcal{O}_X|U_\alpha)^r$
- From this, we can also describe E in terms of its transition functions

$$M_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}\colon (\mathcal{O}_X|U_{\alpha\beta})^r \xrightarrow{\sim} (\mathcal{O}_X|U_{\alpha\beta})^r$$

i.e. "holomorphic function-valued invertible matrices"

- Modulo technical details, these are exactly holomorphic vector bundles, which are exactly "holomorphic $\check{\mathscr{C}}(\mathcal{U}) \to \mathbb{B}\mathrm{GL}_r(\mathbb{C})$"

Coherent analytic sheaves

Oka coherence

- The definition of a coherent sheaf (not even a complex of such objects, just a single sheaf) is somewhat delicate, e.g. Serre (and thus EGA and Stacks) and Hartshorne give different definitions that actually seem to contradict
    - Hartshorne's definition seems to imply that $\mathcal{O}_X$ itself should *always* be a coherent sheaf, but this is not true in general, not even for affine schemes
    - The fact that $\mathcal{O}_X$ is coherent in the holomorphic setting is a very deep theorem of Oka
- The problem boils down to Hartshorne considering only *Noetherian* schemes; in general, the notion of pseudo-coherence from SGA 6 is necessary

Coherent analytic sheaves

Nicer than holomorphic vector bundles

- Four non-definitions:
  - In general, the category of vector bundles is not abelian: can't take (co)kernel — the category of *coherent sheaves* is nice because it contains the category of vector bundles, is abelian, and arises "naturally"
  - In the holomorphic world, "quasi-coherent" isn't well behaved (or even well defined, arguably), but "coherent" is a good analogue
  - Coherence allows us to extend from stalks to neighbourhoods: if $\mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x$ is exact then so too is $\mathscr{F}\,|\,U \to \mathscr{G}\,|\,U \to \mathscr{H}\,|\,U$
  - Historically things go in the other direction: coherence comes from complex geometry, in Cartan's Theorems A and B, which imply *lots* of things on Stein (think "affine") manifolds (e.g. extension of holomorphic functions on closed subsets, a sort of Nullstellensatz, every meromorphic function is the quotient of global holomorphic functions, ...)

Coherent analytic sheaves

Perfect complexes

- (Bounded) cochain complex $L^\bullet$ of sheaves of $\mathcal{O}_X$-modules is
  - **strictly perfect** if each L^i is a (finite) locally-free sheaf
  - **perfect** if it is *locally* quasi-isomorphic to strictly perfect complex
- The relationship between perfectness and coherence is... complicated (see SGA 6) but for this talk, we can think "perfect" whenever we see "coherent"
- (It is in this definition of strictly perfect that we really need *locally* ringed spaces)

Coherent analytic sheaves

Worse than coherent *algebraic* sheaves

- **Fact.** [SGA 6, II, Corollaire 2.2.2.1] Let X be a Noetherian scheme. Then the canonical fully faithful functor $\mathrm{D}^{\mathrm{b}}(\mathsf{Coh}(X)) \hookrightarrow \mathrm{D}(\mathsf{Sh}(X))$ has essential image $\mathrm{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathsf{Sh}(X))$

- **Fact.** [SGA 6, II, Exemples 5.11 (+ Corollaire 2.2.2.1)] Let X be a smooth scheme. Then there is a canonical equivalence of triangulated categories $\mathsf{Perf}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathsf{Coh}(X))$

- **Fact.** [SGA 6, II, Propositions 2.2.7 and 2.2.9] If X is an affine (resp. projective, resp. separated regular Noetherian) scheme, then every perfect complex is quasi-isomorphic to a complex of locally free sheaves (i.e. is strictly perfect)

- **Summary.** In the algebraic world, we can often pick either definition of "coherent complex", and then just work with complexes of locally free sheaves anyway

Coherent analytic sheaves

… but worse than coherent *algebraic* sheaves

- **Fact.** (still true) Every coherent sheaf is perfect
- **Open problem.** Is $\mathrm{D}^{\mathrm{b}}(\mathsf{Coh}(X))$ equivalent to $\mathrm{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathsf{Sh}(X))$ when X is a complex-analytic manifold?
- **Fact.** Complex manifolds have very few holomorphic vector bundles, e.g. [Voisin 2002, Corollary A.5] there exist coherent analytic sheaves that do not admit a resolution by locally free sheaves

Coherent analytic sheaves

… but worse than coherent *algebraic* sheaves

- So we have two problems:

    1. we have to pick which definition of "(complex of) coherent sheaf" to use

    2. we cannot reduce to working with locally free sheaves

- (Bonus problem: even if we *could* work with locally free sheaves, we can't do Chern–Weil theory on them because they essentially never admit holomorphic connections!)

Twisting cochains

Guiding philosophies: a toolbox

Not always true but often useful

1. Things that are *global* in complex-*algebraic* geometry are *local* in complex-*analytic* geometry (e.g. resolutions by locally free sheaves, connections)

2. Things that are *local* can be made *simplicially global*, or: let's use *all* of the Čech nerve

3. Things that are *simplicial* are *dg* (Dold–Puppe/Dold–Kan)

Holomorphic twisting cochains

The inspiration

- A complex of locally free sheaves is the data of:
  - locally, complexes of *free* sheaves
  - isomorphisms on intersections $U_{\alpha\beta}$

Holomorphic twisting cochains

The general idea

- A **twisting cochain** is the data of:
  - locally, complexes of free sheaves
  - quasi-isomorphisms on intersections $U_{\alpha\beta}$
  - homotopies witnessing these quasi-isomorphisms on intersections $U_{\alpha\beta\gamma}$
  - homotopies witnessing the failure of the previous homotopies to commute with the quasi-isomorphisms on intersections $U_{\alpha\beta\gamma\delta}$
  - … (and so on)

Holomorphic twisting cochains

The main theorem

- **Theorem.** [Toledo, Tong 197?] Any coherent sheaf on a complex-analytic manifold can be resolved by a holomorphic twisting cochain
- **Hidden Theorem.** [Wei, Hosgood, 2016] Any "*complex of* coherent sheaves" on a complex-analytic manifold can be resolved by a holomorphic twisting cochain

Holomorphic twisting cochains

Modern theorems

- **Theorem.** [Wei 2016] The dg-category of twisting cochains is a dg-enhancement of the triangulated category of perfect complexes
- **Construction.** [H, Zeinalian 2023] The space of twisting cochains (encapsulating e.g. the notion of weak equivalence of twisting cochains)

Holomorphic twisting cochains

One of the original definitions

- We formalise the "... and so on" by bundling up all the equations together into a single equation in some graded structure:

$$\mathfrak{a} = \bigoplus_{k\in\mathbb{N}} \mathfrak{a}^{k,1-k} \in \operatorname{Tot}^1 \hat{\mathscr{C}}^\bullet(\mathscr{U}, \operatorname{End}^\star(V))$$

such that $\hat{\delta}\mathfrak{a} + \mathfrak{a}^2 = 0$

- We end up recovering the **Maurer–Cartan** equation

Maurer–Cartan and the dg-nerve

The sea of sign errors

- The Maurer–Cartan equation can be seen as the defining equation of the dg-nerve, and …

Theorem 2.5.12 ([GMTZ22a, Corollary 3.5]). *Let $\mathscr{D}$ be a dg-category of cochain complexes of modules, let $X = X_\bullet$ be a simplicial set, and let $\mathscr{L} = \{c_x \in \mathscr{D}\}_{x \in X_0}$ be a labelling of the 0-simplices of $X_\bullet$ by $\mathscr{D}$. Then there is a bijection*

$$\left\{f \in \mathrm{Tot}^1(C^{p,q}(X,\mathscr{D};\mathscr{L})) \mid \mathrm{D}f + f \cdot f = 0\right\} \longleftrightarrow \left\{F : X \to \mathscr{N}^{\mathrm{dg}}(\mathscr{D}) \mid F(x) = c_x \text{ for all } x \in X_0\right\}$$

between Maurer–Cartan elements of $C^{\bullet,\star}(X,\mathscr{D};\mathscr{L})$ and morphisms of simplicial sets from $X_\bullet$ to dg-nerve of $\mathscr{D}$ that agree with the labelling $\mathscr{L}$.

Corollary 2.5.13. *With the notation and hypotheses of Theorem 2.5.12, we have a bijection*

$$\left\{\text{Maurer–Cartan elements of } \mathrm{Tot}^1\left(C^{p,q}(\Delta[n],\mathscr{D};K)\right)\right\} \longleftrightarrow \left\{n\text{-simplices } K \in \mathscr{N}^{\mathrm{dg}}(\mathscr{D})_n\right\}$$

where the n-simplex K defines the labelling $\{i\} \mapsto \mathrm{ver}_i K$.

Green's resolution

A key result in homological algebra

As a consequence/input to Green's construction

- **Corollary. (Semi-strictification)** [Green* 1980] Let $L^\bullet \xrightarrow{\sim} M^\bullet$ be a quasi-isomorphism of bounded complexes of free modules. Then there exist $\widetilde{L}^\bullet$ and $\widetilde{M}^\bullet$ such that

  1. $\widetilde{L}^\bullet \simeq L^\bullet$ and $\widetilde{M}^\bullet \simeq M^\bullet$

  2. $\widetilde{L}^\bullet \cong \widetilde{M}^\bullet$

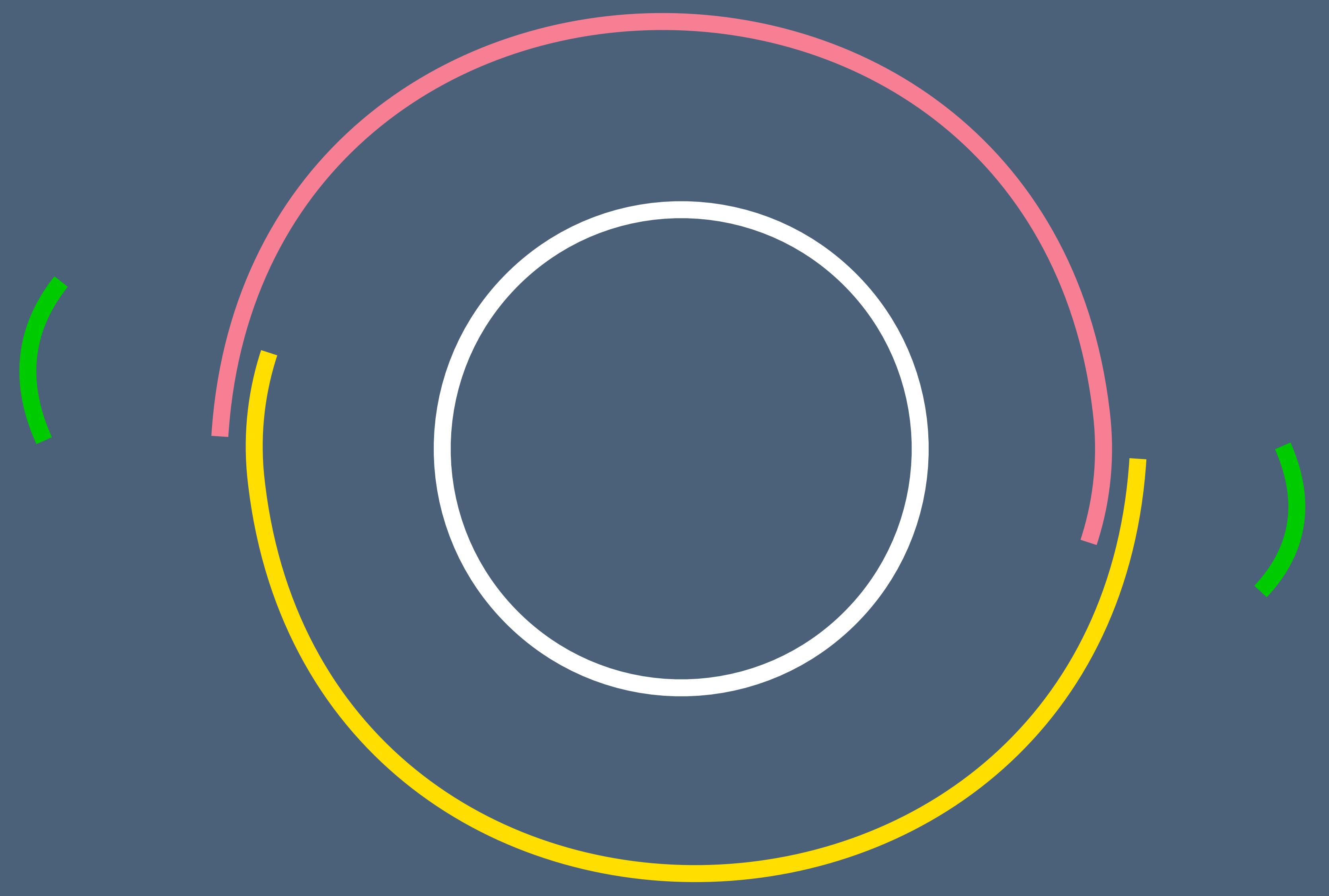
The Čech nerve

My favourite simplicial thing

- **Idea.** Take a cover of a space and stretch it out; if the cover is nice enough then this should be the same as the space

- ("nice enough" = good*, and this implies cofibrant resolution)

$$\ldots \coprod U_{\alpha\beta\gamma} \longleftrightarrow \coprod U_{\alpha\beta} \longleftrightarrow \coprod U_{\alpha}$$



Locally free sheaves on the Čech nerve

In detail, and succinctly

- A sheaf $\mathscr{E}^\bullet$ on a simplicial space $Y_\bullet$ is a family of sheaves $\mathscr{E}^p \in \mathsf{Sh}(Y_p)$ along with morphisms

$$\mathscr{E}^\bullet(\varphi)\colon (Y_\bullet)^*\mathscr{E}^p \to \mathscr{E}^q$$

  for all $\varphi\colon [p] \to [q]$ in Δ, and functorially so; note that we do not require these $\mathscr{E}^\bullet\varphi$ to be isomorphisms or weak equivalences (but in Green's resolution they will be)

- Alternatively [H, 2024] we can just say that they are lifts along $\Delta^{\mathrm{op}} \to \mathsf{Space}$ of the Grothendieck construction of the functor $\mathrm{Sh}\colon \mathsf{Space}^{\mathrm{op}} \to \mathsf{Cat}$

- Alternatively [H 2016] we can define them as lax limit objects of the cosimplicial diagram of categories $\mathrm{Sh}(\check{\mathscr{C}}(\mathscr{U}))$

Green's resolution

"Semi-strictification" of a twisting cochain

$E_\alpha^\bullet$

$E_\beta^\bullet$

$$E_\alpha^\bullet | U_{\alpha\beta} \neq E_\beta^\bullet | U_{\alpha\beta}$$

Green's resolution

"Semi-strictification" of a twisting cochain

$$
\begin{tikzcd}
E^\bullet_\alpha & & & E^\bullet_\beta \\
& E^\bullet_\alpha|U_{\alpha\beta} \arrow[r, dashed] & E^\bullet_\beta|U_{\alpha\beta} &
\end{tikzcd}
$$

Green's resolution

"Semi-strictification" of a twisting cochain

$$\begin{tikzcd}
E^\bullet_\alpha & & & & E^\bullet_\beta \\
& E^\bullet_\alpha|U_{\alpha\beta} \arrow[rr, dashed] \arrow[d, dashed] & & E^\bullet_\beta|U_{\alpha\beta} \arrow[d, dashed] & \\
& \widetilde{E}^\bullet_\alpha|U_{\alpha\beta} \arrow[rr, "\cong", "\varphi_{\alpha\beta}"'] & & \widetilde{E}^\bullet_\beta|U_{\alpha\beta} &
\end{tikzcd}$$

Green's resolution

"Semi-strictification" of a twisting cochain

$$\begin{tikzcd}
 & & & E^\bullet_\beta \arrow[dl, dashed] \arrow[dr, dashed] & & & \\
 & & \widetilde{E}^\bullet_\beta & & \widehat{E}^\bullet_\beta & & \\
 & \widetilde{E}^\bullet_\alpha \arrow[ur, "\varphi_{\alpha\beta}"] & & & & \widehat{E}^\bullet_\gamma \arrow[ul, "\varphi_{\beta\gamma}"'] & \\
E^\bullet_\alpha \arrow[ur, dashed] \arrow[rr, dashed] & & \underline{E}^\bullet_\alpha \arrow[rr, "\varphi_{\alpha\gamma}"'] & & \underline{E}^\bullet_\gamma & & E^\bullet_\gamma \arrow[ul, dashed] \arrow[ll, dashed]
\end{tikzcd}$$

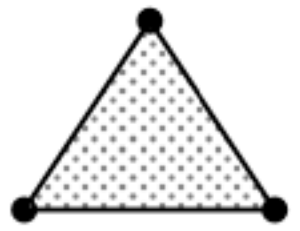
Green's resolution

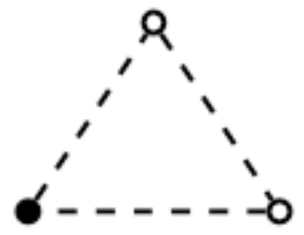
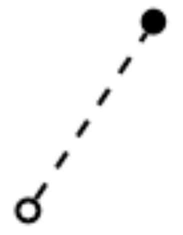
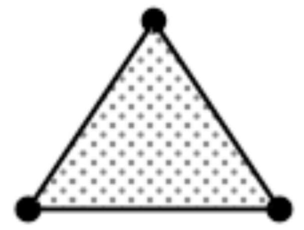
"Semi-strictification" of a twisting cochain

$$
\begin{tikzcd}
 & & & E^\bullet_\beta \arrow[dl, dashed] \arrow[dr, dashed] & & & \\
 & & \widetilde{E}^\bullet_\beta \arrow[dr, dashed] & & \widehat{E}^\bullet_\beta \arrow[dl, dashed] \arrow[ddr, "\varphi_{\beta\gamma}"] & & \\
 & & & \widetilde{\widehat{E}}^\bullet_\beta \arrow[ddr, "\widehat{\varphi}_{\beta\gamma}"] \arrow[dd, phantom, "\Uparrow{=}"] & & & \\
 & \widetilde{E}^\bullet_\alpha \arrow[uur, "\varphi_{\alpha\beta}"] \arrow[dr, dashed] & & & & \widehat{E}^\bullet_\gamma \arrow[dl, dashed] & \\
 & & \underline{\widetilde{E}}^\bullet_\alpha \arrow[uur, "\widetilde{\varphi}_{\alpha\beta}"] \arrow[rr, "\underline{\varphi}_{\alpha\gamma}"'] & & \underline{\widehat{E}}^\bullet_\gamma & & \\
E^\bullet_\alpha \arrow[uur, dashed] \arrow[rr, dashed] & & \underline{E}^\bullet_\alpha \arrow[u, dashed] \arrow[rr, "\varphi_{\alpha\gamma}"'] & & \underline{E}^\bullet_\gamma \arrow[u, dashed] & & E^\bullet_\gamma \arrow[ll, dashed] \arrow[uul, dashed]
\end{tikzcd}
$$

Pair subdivision

A cubical factorisation of the barycentric subdivision

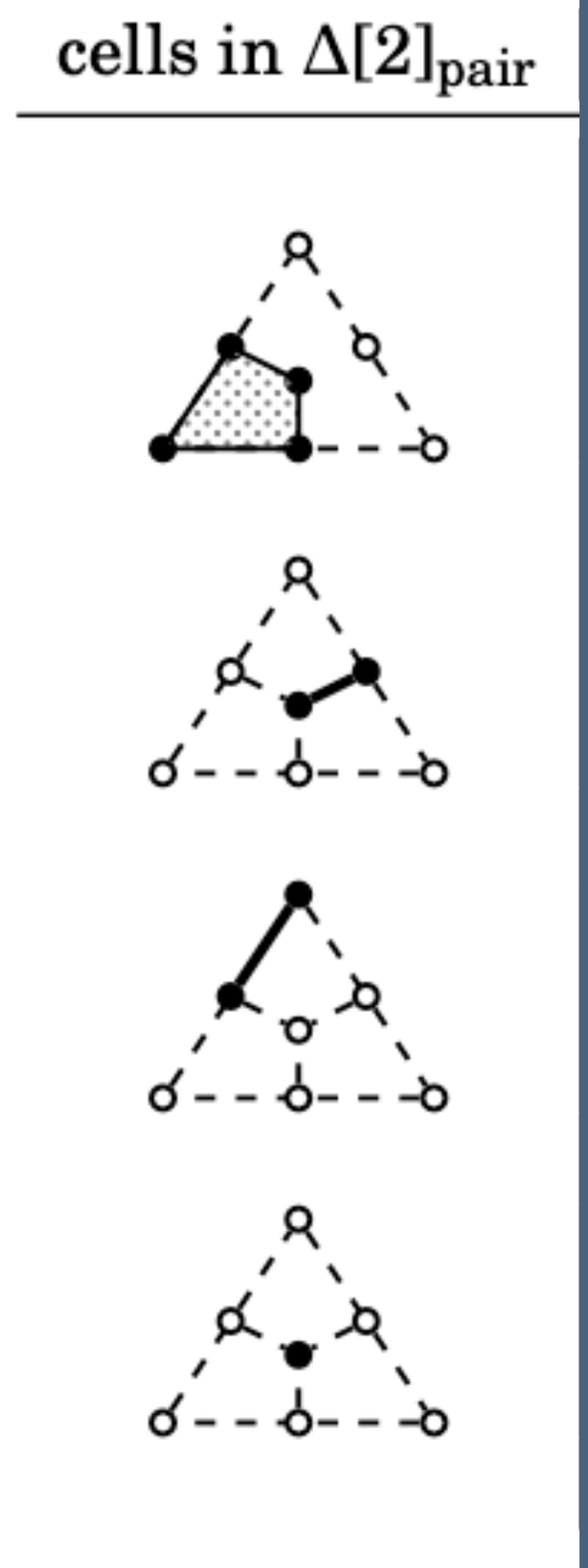
0-simplex of $\Delta[2]_{\mathrm{bary}}$	subset of $[2] = \{0 < 1 < 2\}$	sub-simplex of $\Delta[2]$
	$\{0\} \subseteq [2]$	
	$\{1 < 2\} \subseteq [2]$	
	$\{0 < 1 < 2\} \subseteq [2]$	

cells in $\Delta[2]_{\mathrm{pair}}$	pairs of subsets of $[2] = \{0 < 1 < 2\}$	sub-simplex of $\Delta[p]$
	$\{0\} \subseteq [2]$	
	$\{1 < 2\} \subseteq [2]$	
	$\{1\} \subseteq \{0 < 1\}$	
	$[2] \subseteq [2]$	

Green's resolution

"Semi-strictification" of a twisting cochain

$$\begin{tikzcd}[column sep=small]
 & & & E^\bullet_\beta \arrow[dl, dashed] \arrow[dr, dashed] & & & \\
 & & \widetilde{E}^\bullet_\beta \arrow[dr, dashed] & & \widehat{E}^\bullet_\beta \arrow[dl, dashed] \arrow[ddr, "\varphi_{\beta\gamma}"] & & \\
 & & & \widetilde{\widehat{E}}{}^\bullet_\beta \arrow[ddr, "\widehat{\varphi}_{\beta\gamma}"] & & & \\
 & \widetilde{E}^\bullet_\alpha \arrow[uur, "\varphi_{\alpha\beta}"] \arrow[dr, dashed] & & \arrow[d, phantom, "\Uparrow ="] & & \widehat{E}^\bullet_\gamma \arrow[dl, dashed] & \\
 & & \widetilde{\underline{E}}{}^\bullet_\alpha \arrow[uur, "\widetilde{\varphi}_{\alpha\beta}"] \arrow[rr, "\underline{\varphi}_{\alpha\gamma}"'] & {} & \widehat{\underline{E}}{}^\bullet_\gamma & & \\
E^\bullet_\alpha \arrow[uur, dashed] \arrow[rr, dashed] & & \underline{E}^\bullet_\alpha \arrow[u, dashed] \arrow[rr, "\varphi_{\alpha\gamma}"'] & & \underline{E}^\bullet_\gamma \arrow[u, dashed] & & E^\bullet_\gamma \arrow[ll, dashed] \arrow[uul, dashed]
\end{tikzcd}$$



A Dold–Kan style result

But not exactly… maybe?

- **Theorem.** [Green 1980] Let $\mathscr{F}$ be a coherent analytic sheaf on a complex-analytic manifold with a "nice" Stein cover $\mathscr{U}$. Then we can inductively modify any twisting cochain resolving $\mathscr{F}$ to obtain a "very nice" complex of locally free sheaves on $\check{\mathscr{C}}(\mathscr{U})$ that resolves $\mathscr{F}$.
- In other words, we can package up dg data (quasi-isomorphisms and chain homotopies and … i.e. a Maurer–Cartan element) into simplicial data (a single "global" simplicial object)
- **Open question.** Can we explicitly state an equivalence between twisting cochains and Green complexes as an example of some Dold–Kan statement?

Simplicial twisting cochains

A meeting place for twisting cochains and Green complexes

- Toledo–Tong's summary of Green's thesis [1986] defines a *simplicial twisting cochain* as a sort of ad-hoc common generalisation of twisting cochains and locally free sheaves on the Čech nerve ("simplicial vector bundles")

- Simplest definition: like Green's resolution, but we now label the *full* pair subdivision instead of just the 1-skeleton

- **Question.** What can we say about $\mathscr{T}\mathbf{wist} \hookrightarrow \mathscr{S}\mathbf{Twist} \hookleftarrow \mathscr{G}\mathbf{reen}$ in terms of category/homotopy theory?

Cosimplicial simplicial sets

Totalisation of cosimplicial simplicial sets

Lots of definitions

- Right adjoint to the functor $L\colon \mathsf{sSet} \to \mathsf{csSet}$ given by $L\colon X_\bullet \mapsto X_\bullet \times \Delta[\,\star\,]$
- $\underline{\mathrm{Hom}}_{\mathsf{csSet}}(\Delta[\,\star\,], -\,)$
- $\mathrm{Tot}\, X_\bullet^\star = \mathrm{eq}\left(\prod_{[p]} \mathrm{Hom}_{\mathsf{sSet}}(\Delta[p], X_\bullet^p) \rightrightarrows \prod_{[p]\to[q]} \mathrm{Hom}_{\mathsf{sSet}}(\Delta[p], X_\bullet^q)\right)$
- "Dual to geometric realisation"

Totalisation of cosimplicial simplicial sets

Lots of definitions

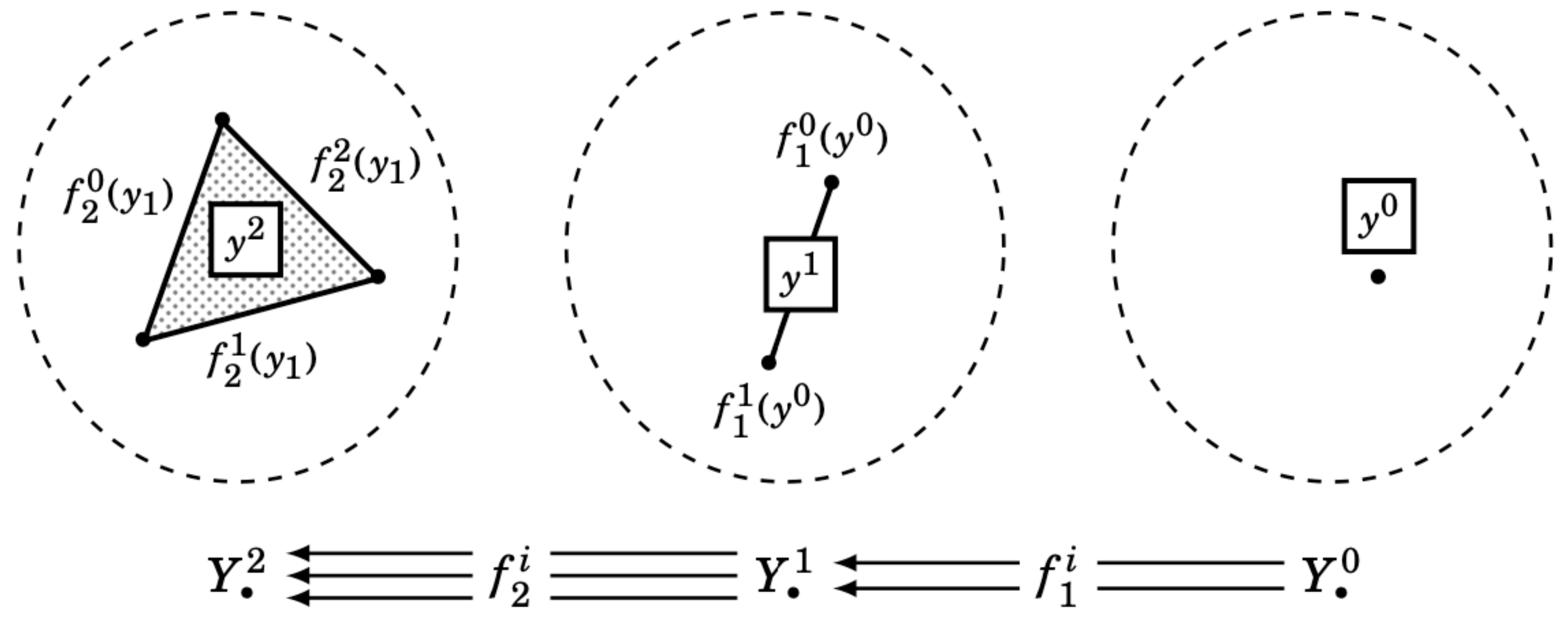


Figure 2.3.i. *Visualising a point $y = (y^0, y^1, y^2, \ldots)$ in the totalisation of a cosimplicial simplicial set $Y_\bullet^\star$. For aesthetic purposes, we have not drawn the codegeneracy maps, nor anything above degree 2.*

Čech totalisation of simplicial presheaves

A method of sheafification* [Glass, Miller, Tradler, Zeinalian 2022]

- The (opposite) of the Čech nerve gives a functor $\check{\mathcal{N}}^{\mathrm{op}}\colon \mathsf{Space}_{\mathcal{U}}^{\mathrm{op}} \to [\Delta, \mathsf{Space}^{\mathrm{op}}]$ and so we can precompose any simplicial presheaf $\mathcal{F}\colon \mathsf{Space}^{\mathrm{op}} \to \mathsf{sSet}$ by this to get a cosimplicial simplicial set $\mathcal{F}(\check{\mathcal{N}}\mathcal{U}_\star)$ whenever we fix some $(X, \mathcal{U}) \in \mathsf{Space}_{\mathcal{U}}$

- **Definition.** The **Čech totalisation** of a simplicial presheaf $\mathcal{F}$ at the cover $\mathcal{U}$ (of a space X) is the simplicial set $\mathrm{Tot}\,\mathcal{F}(\check{\mathcal{N}}\mathcal{U}_\star)$

- This generalises sheafification in terms of taking sections of the espace étalé

- **Lemma.** Čech totalisation sends presheaves of Kan complexes to Kan complexes, and weak equivalences between such presheaves to weak equivalences.

- **Corollary.** [GMTZ 2022] If $\mathcal{F}$ is a presheaf of Kan complexes, then its Čech totalisation is equivalent to its Čech homotopy limit

Čech totalisation of simplicial presheaves

Relating to existing results in the literature

- **Lemma.** [H, Zeinalian 2023] Let $\mathcal{F}$ be a presheaf of dg-categories that sends finite products to coproducts. Then there is a weak equivalent of Kan complexes

$$\mathrm{Tot}\langle \mathcal{N}^{\mathrm{dg}}\mathcal{F}(\check{\mathcal{N}}\mathcal{U})\rangle \simeq \langle \mathcal{N}^{\mathrm{dg}}\,\mathrm{Tot}\,\mathcal{F}(\check{\mathcal{N}}\mathcal{U})\rangle$$

  where on the left we take the totalisation of cosimplicial simplicial sets, and on the right we take the totalisation of cosimplicial dg-categories

- **Corollary.** Working with simplicial presheaves recovers analogous results about homotopy limits of dg-categories of chain complexes in the language of presheaves of dg-categories [Block, Holstein, Wei 2017]

Encoding structures in simplicial presheaves

Vector bundles

- $\mathscr{B}\mathrm{un}_{\mathrm{GL}_r(\mathbb{R})} = \mathrm{Tot}((\check{\mathscr{N}}^{\mathrm{op}})^* \mathscr{N}\mathbb{B}y)(\mathrm{GL}_r(\mathbb{R}))\colon \mathsf{Space}_{\mathscr{U}} \to \mathsf{sSet}$

- **Theorem.** ["folklore"] $\pi_0(\mathscr{B}\mathrm{un}_{\mathrm{GL}_r(\mathbb{R})}(X, \mathscr{U}))$ consists of isomorphism classes of principal $\mathrm{GL}_r(\mathbb{R})$-bundles on X (with $\mathscr{U}$ trivialising); $\pi_1(\mathscr{B}\mathrm{un}_{\mathrm{GL}_r(\mathbb{R})}(X, \mathscr{U}), [E])$ is the gauge group $\mathrm{Aut}(E)$ of E; higher homotopy groups are zero.

Encoding structures in simplicial presheaves

Green complexes, twisting cochains, and simplicial twisting cochains

- Aim: generalise $\mathrm{Tot}\langle \mathscr{N}\mathsf{Free}(\check{\mathscr{N}}\mathscr{U}_\star)\rangle$, which is the space of locally free sheaves
- Inspired by Dold–Kan, we can try two things [H, Zeinalian 2023]

  1. replace $\mathscr{N}$ by $\mathscr{N}^{\mathrm{dg}}$ $\rightsquigarrow$ allow gluing by *quasi*-isomorphisms and higher homotopies $\rightsquigarrow$ twisting cochains

  2. do "something simplicial" $\rightsquigarrow$ label the pair subdivision $\rightsquigarrow$ Green complexes

- Bonus fact: if we do *both* then we get simplicial twisting cochains

Summary / unanswered questions

Can we do everything pre-geometry? In a "neat" way?

- We can define three simplicial presheaves on the category of locally ringed spaces such that their Čech totalisation recovers twisting cochains, Green complexes, and simplicial twisting cochains (respectively)

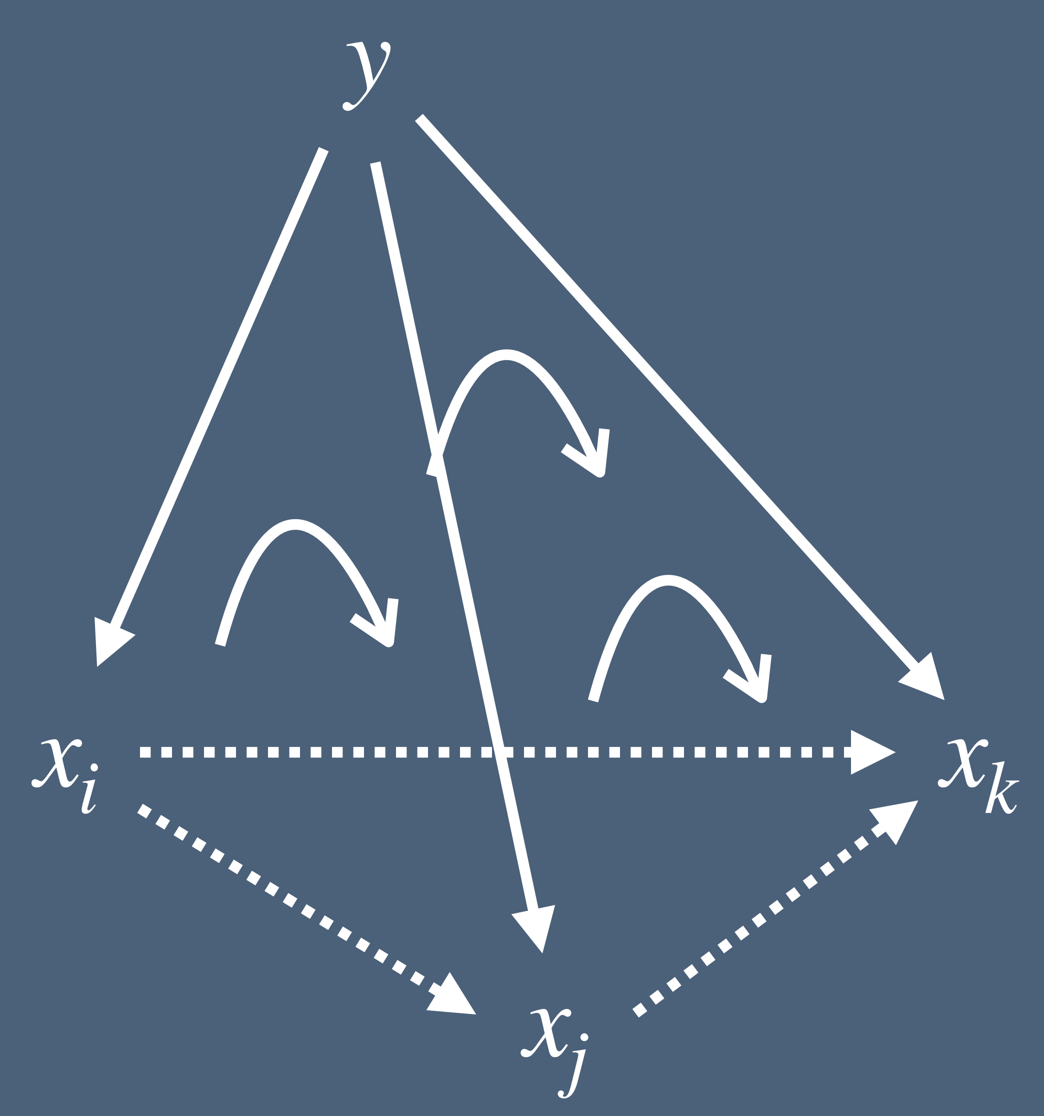
- We can show an equivalence on π_0 *before* Čech totalisation ("pre-geometry")

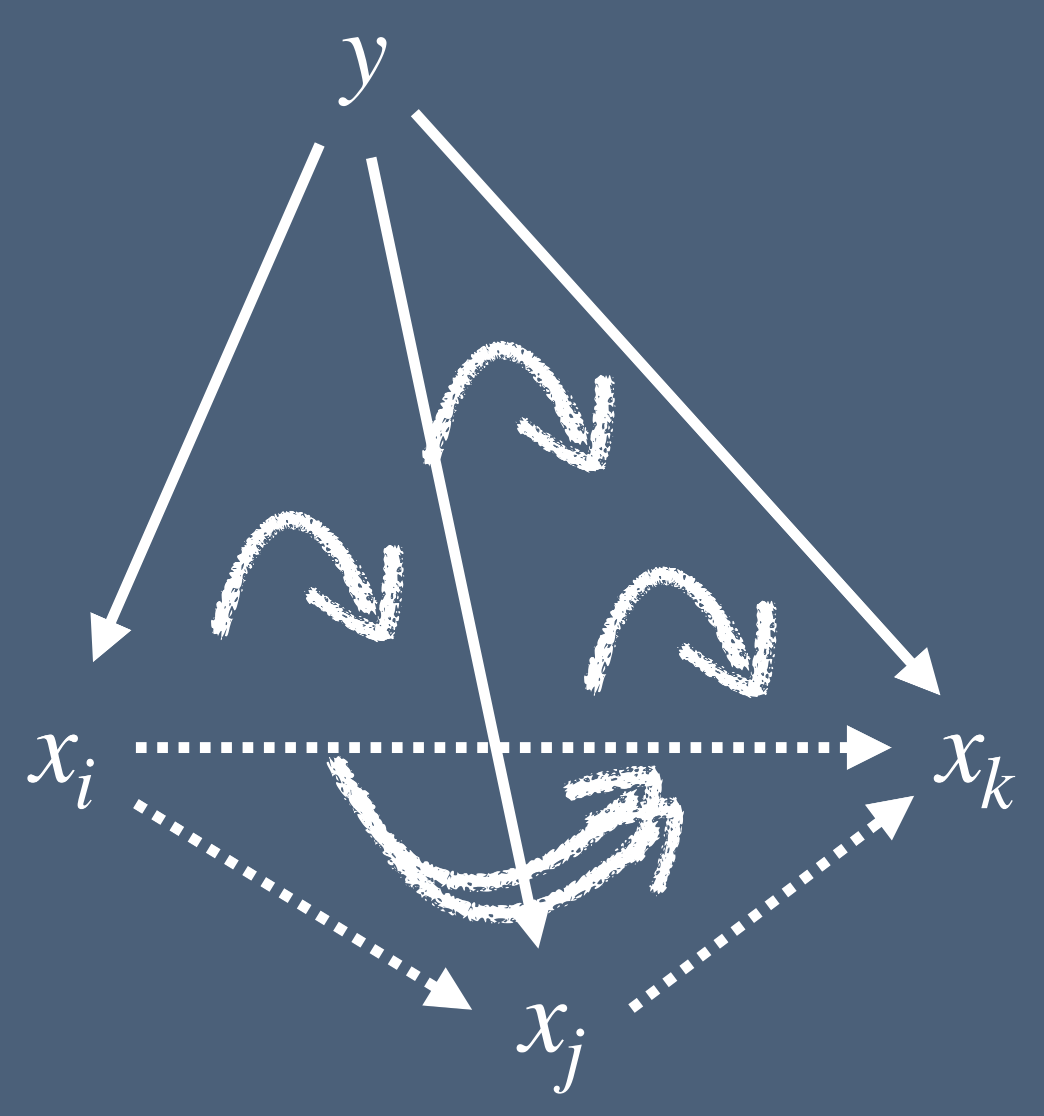
- ... but doing anything else is hard because the construction $\mathscr{G}\mathbf{reen}$ is so ad-hoc, with lots of combinatorics to keep track of

Homotopy limits

Some intuition for homotopy limits

"Homotopy-universal homotopy-cone"





Homotopy equalisers

A particularly simple (and relevant) type of homotopy limit

- 1-categorical: we have $X \rightrightarrows Y$ and want to find Z such that the images are isomorphic

$$
\begin{tikzcd}
{} \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\triangle"] \\
X \times X \arrow[r, "f \times g"'] & Y \times Y
\end{tikzcd}
$$

Homotopy equalisers

A particularly simple (and relevant) type of homotopy limit

- Homotopical: we have $X \rightrightarrows Y$ and want to find Z such that the images are *homotopic*

$$
\begin{tikzcd}
 & Y \cong Y^{\Delta[0]} \arrow[d, "\wr"'] \arrow[dd, bend left=40] & \\
{} \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner_h" very near start] & Y^{\Delta[1]} \arrow[d, "\partial"] \arrow[r, phantom, "\triangle"] & {} \\
X \times X \arrow[r, "f \times g"'] & Y \times Y \cong Y^{\partial\Delta[1]} &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
{} \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner" very near start] & Y^{\Delta[1]} \arrow[d, "\partial"] \\
X \times X \arrow[r, "f \times g"'] & Y \times Y \cong Y^{\partial\Delta[1]}
\end{tikzcd}
$$

Fat Delta and Bousfield–Kan

Proving that the totalisation (sometimes) computes the homotopy limit

- As an equaliser, Tot uses $\Delta[p]$ as a cotensor

$$\mathrm{Tot}\, X_\bullet^\star = \mathrm{eq}\Big(\prod_{[p]} \mathrm{Hom}_{\mathsf{sSet}}(\Delta[p], X_\bullet^p) \rightrightarrows \prod_{[p]\to[q]} \mathrm{Hom}_{\mathsf{sSet}}(\Delta[p], X_\bullet^q)\Big)$$

  and $\mathbf{holim}$ can be written in the same way [Hirschhorn 2003] but replacing Δ by the **fat** simplex, defined by $\boldsymbol{\Delta}[p] = \mathcal{N}(\Delta/[p])$

- Both Δ and $\boldsymbol{\Delta}$ are cofibrant replacements of $*$ but not *fibrant* cofibrant, so Bousfield–Kan built a map directly between them to witness the weak equivalence

Loose simplices

*in progress

Back to simplicial twisting cochains

The nerve inside the dg-nerve and the equaliser inside the homotopy equaliser

$$
\begin{tikzcd}
\mathscr{V}\mathrm{ectBun} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{T}\mathrm{wist} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] & \mathrm{eq} = \mathrm{Tot} \arrow[d, leftrightarrow] \\
\mathscr{G}\mathrm{reen} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{S}\mathrm{Twist} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\Delta[1]} \arrow[d] & \mathrm{hoeq} = \mathrm{Tot}_1 \\
\mathscr{N} \arrow[r] & \mathscr{N}^{\mathrm{dg}} \arrow[r] & Y \times Y & \\
\cong \arrow[r, leftrightarrow] & \simeq & &
\end{tikzcd}
$$

Back to simplicial twisting cochains

The nerve inside the dg-nerve and the equaliser inside the homotopy equaliser

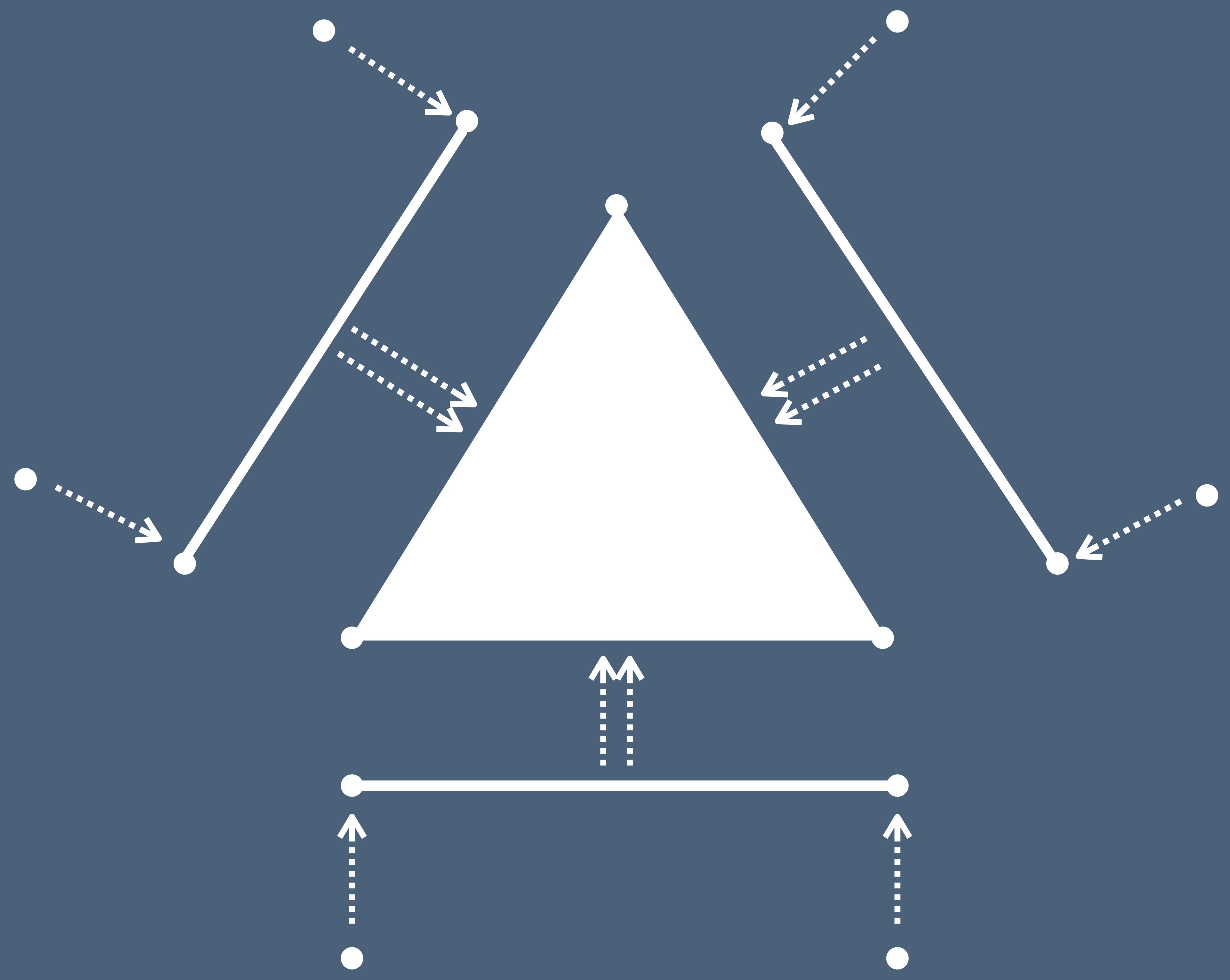
$$
\begin{tikzcd}
\mathscr{V}\mathrm{ectBun} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{T}\mathrm{wist} \arrow[r] \arrow[d, "\simeq ?"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
\mathscr{G}\mathrm{reen} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{S}\mathrm{Twist} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\Delta[1]} \arrow[d] \\
\mathscr{N} \arrow[r] & \mathscr{N}^{\mathrm{dg}} \arrow[r] & Y \times Y \\
\cong \arrow[r, leftrightarrow] & \simeq &
\end{tikzcd}
$$

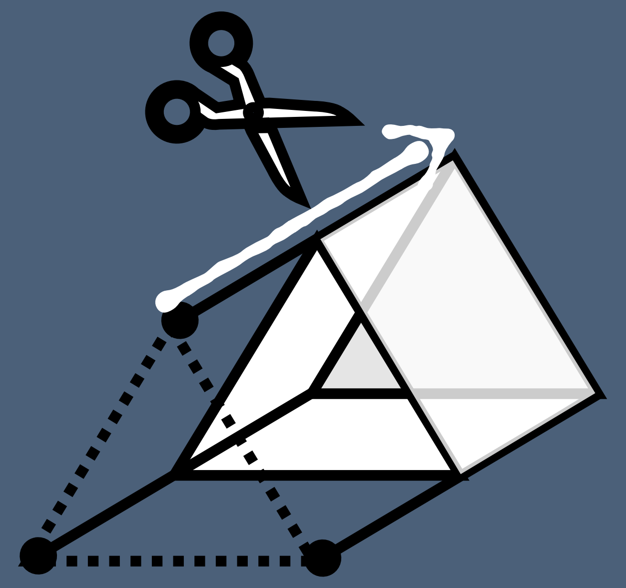
$$
\begin{tikzcd}
\mathrm{eq} = \mathrm{Tot} \arrow[d, leftrightarrow] \\
\mathrm{hoeq} = \mathrm{Tot}_1
\end{tikzcd}
$$

Idea: compare Tot_1 to **holim**
just like we do for Tot

The reoccurring picture

Hopefully somewhat familiar now

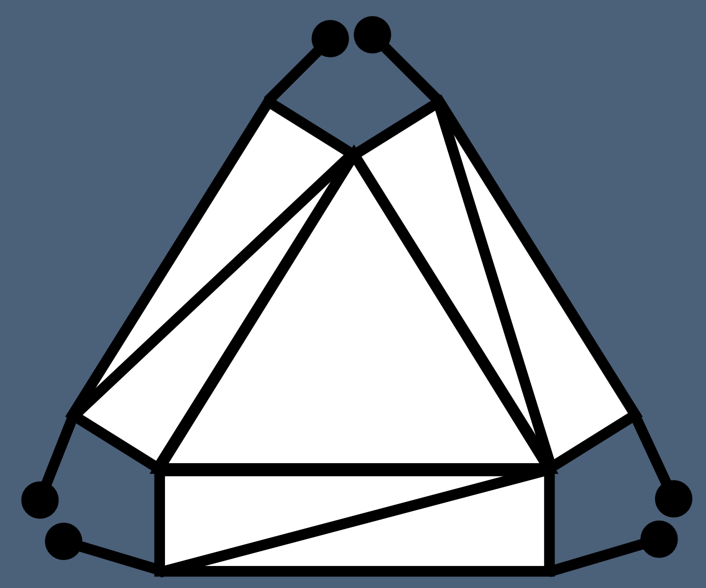




$$X_\bullet \simeq \operatorname{colim} \mathrm{sk}_n X_\bullet \simeq \operatorname{hocolim} \mathrm{sk}_n X_\bullet$$

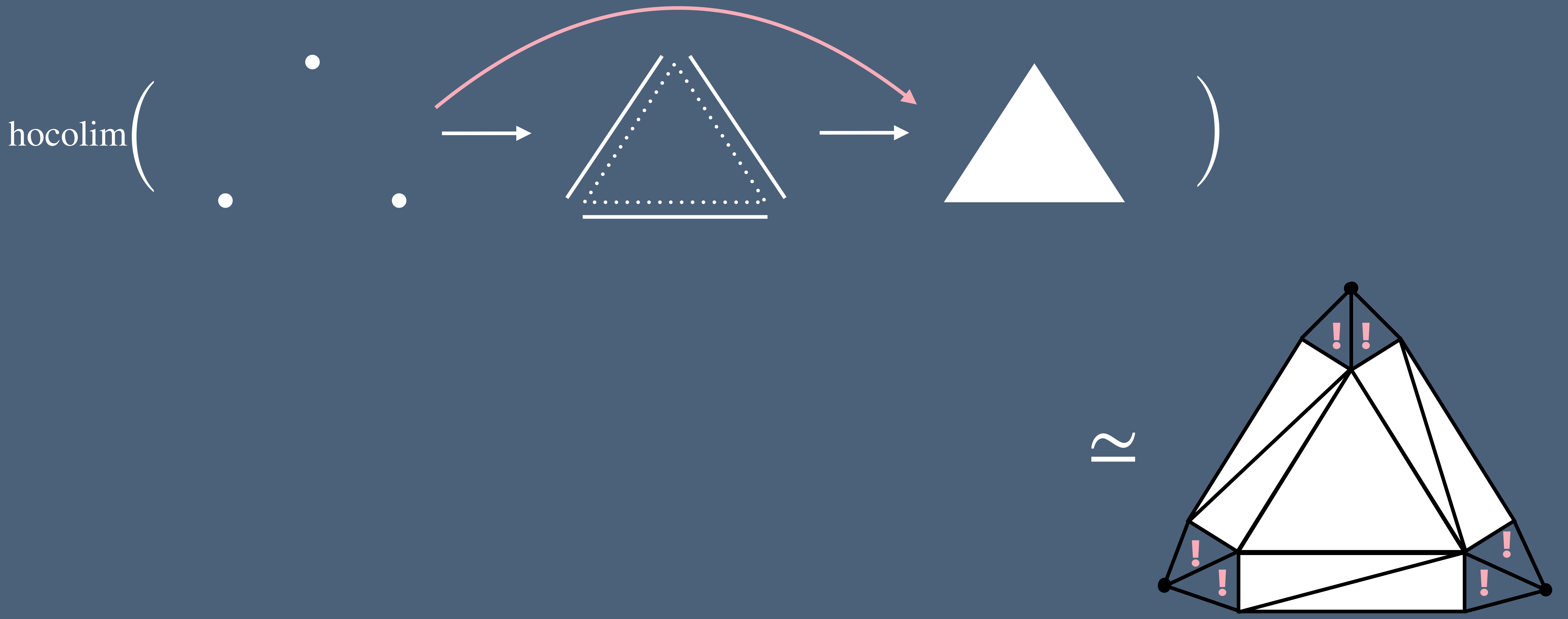
this is a specific model for **this**

(for $X_\bullet = \Delta^p_\bullet$)



Homotopy colimit of skeletal filtration

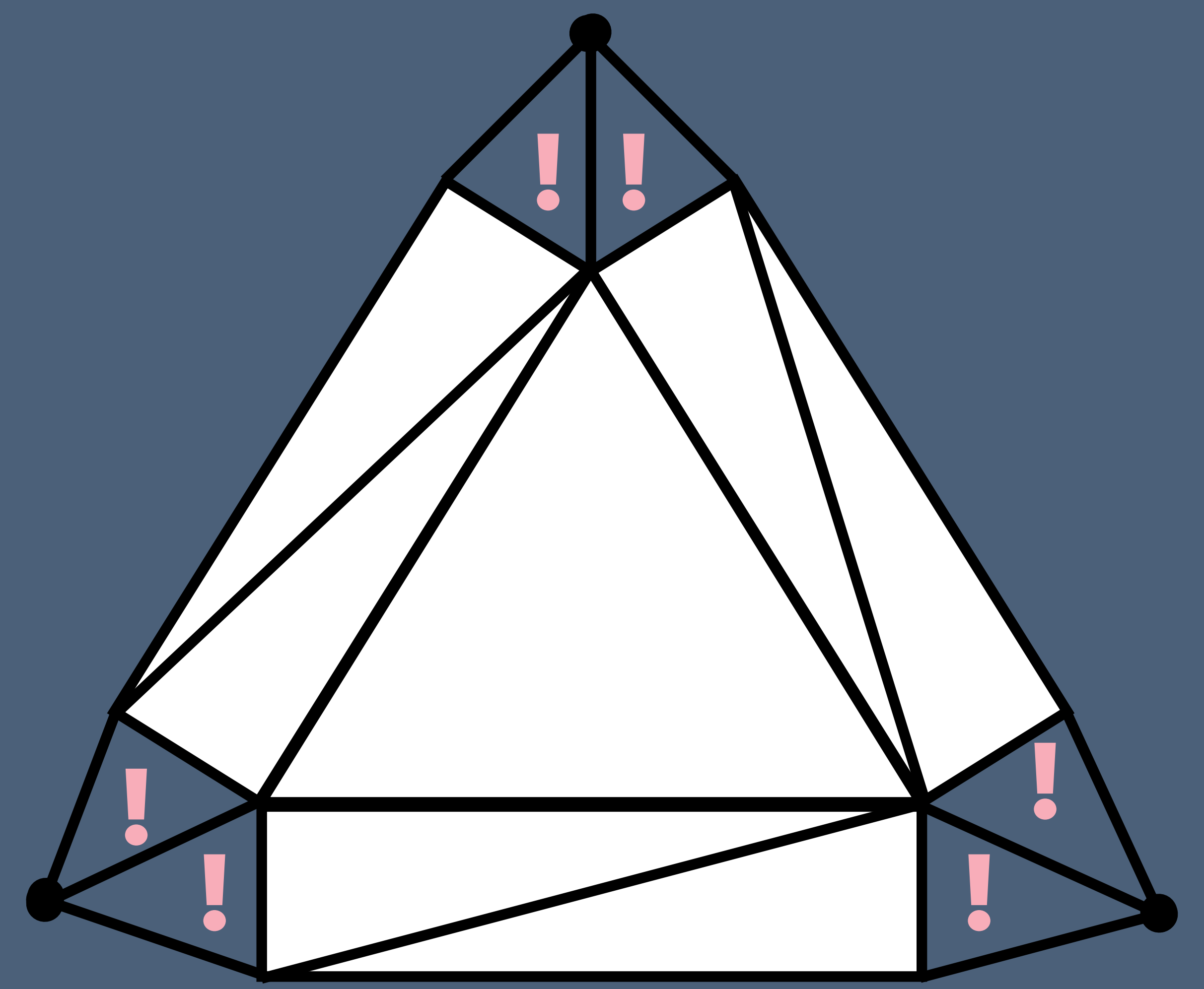
We get non-trivial 2-dimensional data from composites



"Bounded-homotopy" colimit

Only allowing homotopical data up to dimension 1

$$
\begin{tikzcd}
 & & & & E_\beta^\bullet \arrow[dl, dashed] \arrow[dr, dashed] & & & & \\
 & & & \widetilde{E}_\beta^\bullet \arrow[dr, dashed] & & \widehat{E}_\beta^\bullet \arrow[dl, dashed] \arrow[ddrr, "\varphi_{\beta\gamma}"] & & & \\
 & & & & \widetilde{\widehat{E}}_\beta^\bullet \arrow[ddrr, "\widehat{\varphi}_{\beta\gamma}"] \arrow[dd, phantom, "\Uparrow ="] & & & & \\
 & \widetilde{E}_\alpha^\bullet \arrow[uurr, "\varphi_{\alpha\beta}"] \arrow[dr, dashed] & & & & & & \widehat{E}_\gamma^\bullet \arrow[dl, dashed] & \\
 & & \widetilde{\underline{E}}_\alpha^\bullet \arrow[uurr, "\widetilde{\varphi}_{\alpha\beta}"] \arrow[rrrr, "\underline{\varphi}_{\alpha\gamma}"'] & & & & \widehat{\underline{E}}_\gamma^\bullet & & \\
E_\alpha^\bullet \arrow[uur, dashed] \arrow[rr, dashed] & & \underline{E}_\alpha^\bullet \arrow[u, dashed] \arrow[rrrr, "\varphi_{\alpha\gamma}"'] & & & & \underline{E}_\gamma^\bullet \arrow[u, dashed] & & E_\gamma^\bullet \arrow[uul, dashed] \arrow[ll, dashed]
\end{tikzcd}
$$

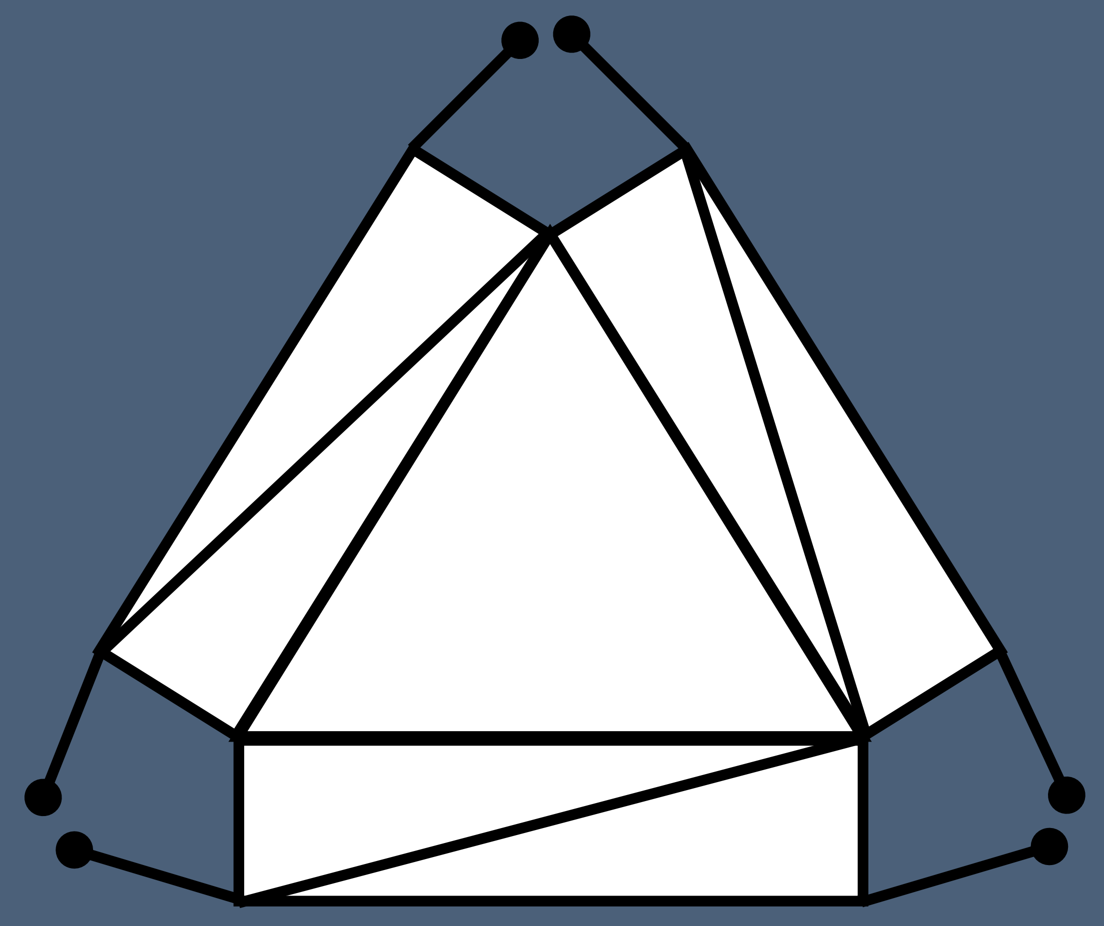


"Bounded-homotopy" colimit

Only allowing homotopical data up to dimension 1

- Solution: come up with a "loose" skeleton, and take the homotopy colimit of these in a specific way

- **Claim.** This gives a formal definition of the cosimplicial simplicial set called the "extruded simplex"



Sitting between holim and Tot

Are simplicial twisting cochains twisting cochains?

- The extruded simplex sits in between the simplex and the fat simplex
- So if we use it as a cotensor then we can place $\mathbf{Tot}_1$ in between $\mathbf{Tot}$ and $\mathbf{holim}$ and study $\mathbf{Tot} = \mathscr{T}\mathbf{wist} \to \mathscr{S}\mathbf{Twist} = \mathbf{Tot}_1$
- **Conjecture.** Some sort of squeezing/sandwich theorem shows that $\mathscr{T}\mathbf{wist} \simeq \mathscr{S}\mathbf{Twist}$

Back to simplicial twisting cochains

The nerve inside the dg-nerve and the equaliser inside the homotopy equaliser

$$\begin{tikzcd}
 & & \mathsf{Coh} & & & \\
 & \mathscr{V}\mathrm{ectBun} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{T}\mathrm{wist} \arrow[u, two heads] \arrow[r] \arrow[d, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] & & \mathrm{eq} = \mathrm{Tot} \arrow[d, leftrightarrow] \\
\mathsf{CCoh} & \mathscr{G}\mathrm{reen} \arrow[l, two heads] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{S}\mathrm{Twist} \arrow[l, bend right, dotted] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\Delta[1]} \arrow[d] & & \mathrm{hoeq} = \mathrm{Tot}_1 \\
 & \mathscr{N} \arrow[r] & \mathscr{N}^{\mathrm{dg}} \arrow[r] & Y \times Y & & \\
 & \cong \arrow[r, leftrightarrow] & \simeq & & &
\end{tikzcd}$$

Thank you